Solutions Set #7: Radioactive Decay and Half-Life

1. The rubidium nuclide ${}_{37}\mathrm{Rb}^{87}$, which emits an electron, is used to determine the age of rocks. When the rock formed, there was Rb in the rocks, but no strontium. When the chemical analysis of an existing rock was done it is found to have a number ratio of ${}_{38}\mathrm{Sr}^{87}$ to ${}_{37}\mathrm{Rb}^{87}$= .018.

a) Explain where the strontium came from.

If the Rb were to undergo beta decay, it would still have a B number of 87, but its Z number would increase from 37 to 38. Those numbers describe the transition from Rb to Sr, so I hypothesize that the Sr in the rock came from the beta decay of Rb.

b) Will this ratio remain constant? Why or why not?

Over time, since there was no Sr to start with but the overall number of nuclei will stay the same, the ratio will continue to increase.

c) Determine the age of the fossils that are contained in this rock.

To solve this problem, I need to solve three equations for the three unknowns in the problem. The three unknowns are the amount of time that has passed, the ratio of the amount of Rb now to the initial amount of Rb, and the amount of Sr now.

I know that the total number of nuclei has not changed.

$$N_R(\text{now}) + N_S(\text{now}) = N_R(\text{initially}) + N_S(\text{initially}) = N_R(\text{initially}) \tag{1}$$

I know the number ratio of nuclei now.

$$\frac{N_S(\text{now})}{N_R(\text{now})} = 0.018 \tag{2}$$

I also know that the Rb has a halflife of $\tau = 4.75 \times 10^{10}$ years:

$$N_R(\text{now}) = N_R(\text{initially}) \times e^{-\frac{t}{\tau}\ln 2} \tag{3}$$

where t is the amount of time that has passed from "initially" to "now". I can plug Equation 1 into Equation 2 to get

$$\frac{N_R(\text{initially}) - N_R(\text{now})}{N_R(\text{now})} = \frac{N_R(\text{initially})}{N_R(\text{now})} - 1 = 0.018 \tag{4}$$

Now I can plug Equation 3 into Equation 4 to get

$$e^{\frac{t}{\tau}\ln 2} - 1 = 0.018 \rightarrow e^{\frac{t}{\tau}\ln 2} = 1.018 \tag{5}$$

Take the natural log of both sides to get:

$$\frac{t}{\tau}\ln 2 = 0.01784 \rightarrow t = 0.02574 \times \tau \tag{6}$$

Since the half life is $\tau = 4.75 \times 10^{10}$ years, then the time elapsed from then until now is 1.22×10^9 years.

2. The average ratio of C^{14}/C^{12} in The Shroud of Turin was measured to be 0.92 ± 0.04 of the ratio one would find in a living organism. To 95% confidence, when was the Shroud of Turin made?

I look up the decay of C^{14} and find that it has a half-life of 5730 yr. Since the amount of C^{12} remains constant, the ratio of C^{14}/C^{12} now is also the same as the ratio of C^{14} now to C^{14} at the time the sheep was sheared to make the cloth, since living organisms maintain a roughly constant ratio.

This means I can find the age according to

$$0.92 = e^{-\frac{t}{5730}\ \ln 2} \rightarrow -\frac{t}{5730}\ \ln 2 = \ln 0.92 \tag{7}$$

which means

$$t = -\frac{\ln 0.92}{\ln 2}\ 5730\ \text{yr} = 689.29\ \text{yr}. \tag{8}$$

Dealing with the uncertainty can be tricky. Perhaps the simplest way, which is not mathematically rigorous, is to subtract the 0.08 from 0.92 and then use 0.84 to estimate the age. That age would then represent an estimate of the lower end of the uncertainty range. If you did that, you would get a 95% limit of 1400 yr (to two sig figs).

Another way to do it is to take the derivative, using the formula

$$\sigma_t = \frac{\partial t}{\partial f}\ \sigma_f, \tag{9}$$

where f is the fraction of the isotopes. The derivative of $\ln f$ is $1/f$, so Equation 9 becomes

$$\sigma_t = \frac{5730\ \text{yr}\ \times \sigma_f}{\ln 2 \times f} = \frac{5730\ \text{yr}\ \times\ 0.04}{\ln 2\ \times\ 0.92} = 359\ \text{yr}. \tag{10}$$

To state that as a final answer, we need to round the uncertainty to a single digit and then round the answer to the same precision. This yields 700 ± 700 yr (two sigma uncertainty, or 95%). Again, the 95% limit is about 1400 yrs ago.

That puts the year when the wool was shorn somewhere between now and the year 600 CE. So, either way, highly unlikely to be 2000 years old.

3. Data show that when Uranium atoms are formed in the explosions of massive stars, the number ratio of U^{235} to U^{238} falls in the range of 1.16 and 1.34. (i.e. for every 100 U^{238} atoms there are 116 to 134 U^{235} atoms.) However, when we examine the uranium in the Earth today, we find that ratio to be 0.00723 (i.e. for every 100,000 U^{238} atoms there are 723 U^{235} atoms.). Use these numbers to estimate how long ago the star exploded, out of whose ashes the solar system formed.

Consider the time-dependent fraction of the number of Uranium nuclei in each of these two isotopes. The Chart of the Nuclides tells me that U^{235} has a half-life of $\tau_{235} = 7.038 \times 10^8$ yr, while U^{238} has a half-life of $\tau_{238} = 4.468 \times 10^9$ yr. Let $x \equiv N_{235}/N_{238}$ be the original ratio of isotopes back when the original star exploded (in the range of 1.16 and 1.34), and $y \equiv 0.00723$ (the ratio now at time t). I can then use the general formula for radioactive decay to get

$$y = \frac{\text{U}^{235}}{\text{U}^{238}}(t) = \frac{N_{235}}{N_{238}} \frac{e^{-\ln 2 \frac{t}{\tau_{235}}}}{e^{-\ln 2 \frac{t}{\tau_{238}}}} = \frac{N_{235}}{N_{238}} \, e^{-t \ln 2 (\frac{1}{\tau_{235}} - \frac{1}{\tau_{238}})} = x \, e^{-t \ln 2 (\frac{1}{\tau_{235}} - \frac{1}{\tau_{238}})}. \tag{11}$$

Now, define a new variable a such that $\frac{1}{a} \equiv \frac{1}{\tau_{235}} - \frac{1}{\tau_{238}}$ and this becomes

$$y = x \, e^{-\frac{t}{a} \ln 2} = \frac{x}{2^{t/a}} \rightarrow \frac{t}{a} \, \ln 2 = \ln \frac{x}{y} \rightarrow t = a \, \frac{\ln \frac{x}{y}}{\ln 2} \tag{12}$$

Plugging in the numbers, $a = 8.3539 \times 10^8$ yr. For the two values of x that mark the boundaries of the range, we get $x_1/y = 160.44$ and $x_2/y = 185.34$. If I plug these numbers into Equation 12, I get $t(x_1) = 6.1200 \times 10^9$ yr and $t(x_2) = 6.2939 \times 10^9$ yr. This means that the star from which the solar system got its Uranium blew up between 6.12 and 6.29 billion years ago.

Of course, life is rarely that simple, and this is merely a lower limit to the age, because the material that made up the solar system did not come from just one star. However, this does clearly set the time scale for how old the material in the solar system must be. If you were to do it more thoroughly, and allow for uranium to come from other stars, the time gets pushed back to about 10 billion years.

4. An old-fashioned silver quarter was irradiated with slow neutrons. These neutrons are captured by the silver nuclei in the quarter. The quarter was then placed close to a Geiger counter and the following count rates (in counts/sec) were recorded as a function of time.
a) Using the table of nuclides that has a link on the class web page, determine the reactions that could happen if the silver nuclei in the quarter absorbed a neutron. Are the products of the neutron capture stable?

According to the web page, silver exists in nature in roughly equal abundances of both Ag^{107} and Ag^{109}. So we can hypothesize that a random sample of silver will contain roughly equal amounts of each isotope. We will be able to test this hypothesis. If these nuclei were to absorb a neutron, they would become Ag^{108} and Ag^{110}. With a half-life of 2.37 min, 97% of the Ag^{108} will undergo beta decay to Cd^{108}, but more than 99% of the Ag^{110} beta-decays to Cd^{110} with a half-life of 24.6 s! That's not stable.
b) Determine the half-life of the decay of the silver nuclides produced when the silver in the quarter absorbed a neutron. Use the data to determine your answer. After you have your answers, look up the half-lives on the table of nuclides on the web and compare.

I entered the table into Graphical Analysis and plotted the natural log of the counts vs. time. The resulting graph is shown in Figure 1. I estimated the uncertainties by taking the square root of the number of counts, which I deduced by assuming all activity measurements were averaged over 20 s intervals, and then dividing by 20 s to get back into activity units.

There are clearly two distinct decays going on. One decay has a much shorter half-life than the other, so by the time 200 s have gone by, whatever was undergoing the first decay is pretty much gone, but the longer-lived decay is still happening. So far, this is consistent with the hypothesis that we are seeing the decay of a mix of Ag-108 and Ag-110, but we need to dig deeper.

I therefore fit a straight line to the latter part of the curve to get the long-lived decay by itself. This yields a best-fit line of

$$y = (-0.0049 \pm 0.0001 \text{ s}^{-1})t - 0.74 \pm 0.05. \tag{13}$$

I can convert Equation 13 back into activity by putting both sides in the exponent of e. Then I subtracted this exponential curve from all the measured activity (which presumably contains counts from both isotopes) to get the short-lived decay by itself. Then I take the log of that difference to get the straight line. This result is shown in Figure 2. I have propogated the uncertainties through the algebra using the standard formula.

When I fit the first nine points in Figure 2 (the points before 200 s, when the error bars are still reasonably small) to a straight line, I got

$$y = (-0.031 \pm 0.002 \text{ s}^{-1})t - 0.1 \pm 0.2. \tag{14}$$

To get the half-life of each decay, I divide the natural log of 2 by the best-fit slope, and get 140 ± 3 s and 23 ± 1 s. These values are not significantly different from the half-lives reported in the table (2.37 min is 142 s, which is less than one sigma from my result, and 24.6 s is less than two sigma from my result), so I conclude that my hypothesis is not wrong: I have observed the decay of Ag^{108} and Ag^{110}.

An alternate way to check this (if you don't want to have to calculate uncertainties for your measured half-lives) is to convert the half-lives given in the web chart into predictions for what the slope of the semilog graph would be, if the hypothesis is not wrong. If you do that, you get 0.00487445 s^{-1} and 0.02818 s^{-1}. As before, the slower decay is not different to less than one sigma, and the faster decay is not different to less than two sigma. Either way, the hypothesis is not wrong.

5. Just looking at the plot of world population vs. time shows a curve that looks very much like an exponential growth curve. However, plotting the log of the population vs. time reveals a more complex story. In Figure 3, it appears as though the data are consistent with *two* exponential growth curves. Prior to the early 20th century, all the data are consistent with a doubling time (opposite of half-life) of 151 ± 6 yr, but then the rate increases, becoming consistent with a new doubling time of 35.56 ± 0.05 yr. Something happened in the first half of the 20th century that drastically affected the way the human population grows on this planet. According to Rex, who did a lot of work on this some years back, the key element is improvements in medical technology, in particular sterilization and the wide availability of antibiotics. Without that, if the trend from the three *hundred* previous years had continued, there would be more like 2.6 billion people on the planet today.

Interestingly enough, this pattern seems to be in the process of changing again. At any rate, if you extrapolate the second, faster function to 2009, you would predict 9 billion people on the planet this year. If you look closely at the graph, the last point lies quite a bit under the line, so the slowdown appears to have begun in the last ten years or so.

Also fun, as long as we're playing with numbers, is to try to predict with the first curve when there were just a few people on earth. That goes back to about 2000 BC, and since people have been around longer than that (even the creationists think so), this curve cannot be accurate over that much time.

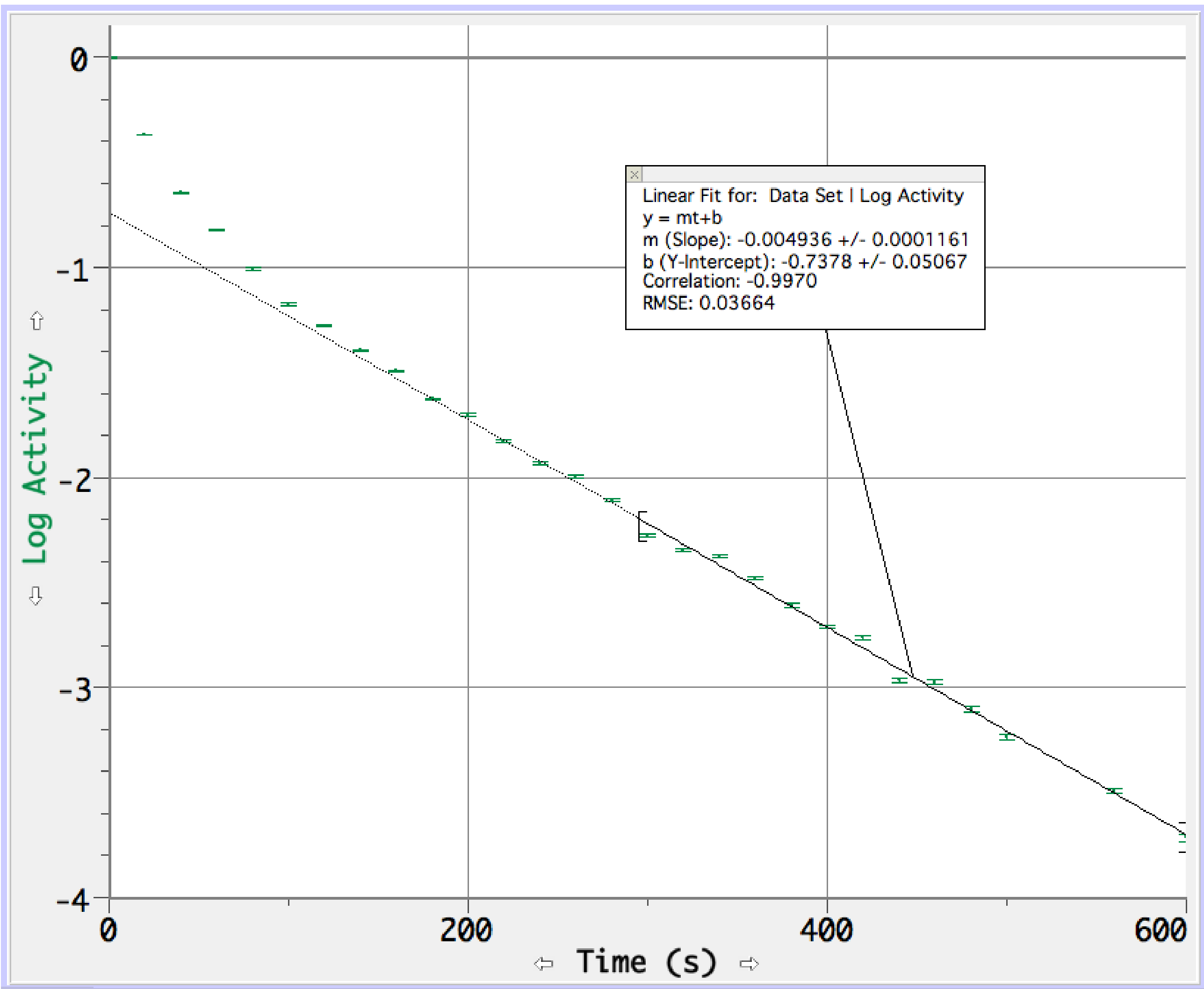

Figure 1: Natural log of Activity (normalized) vs. time (in seconds) from an irradiated silver quarter. Note that there are two straight line segments, with a crossover time of about 200 s.

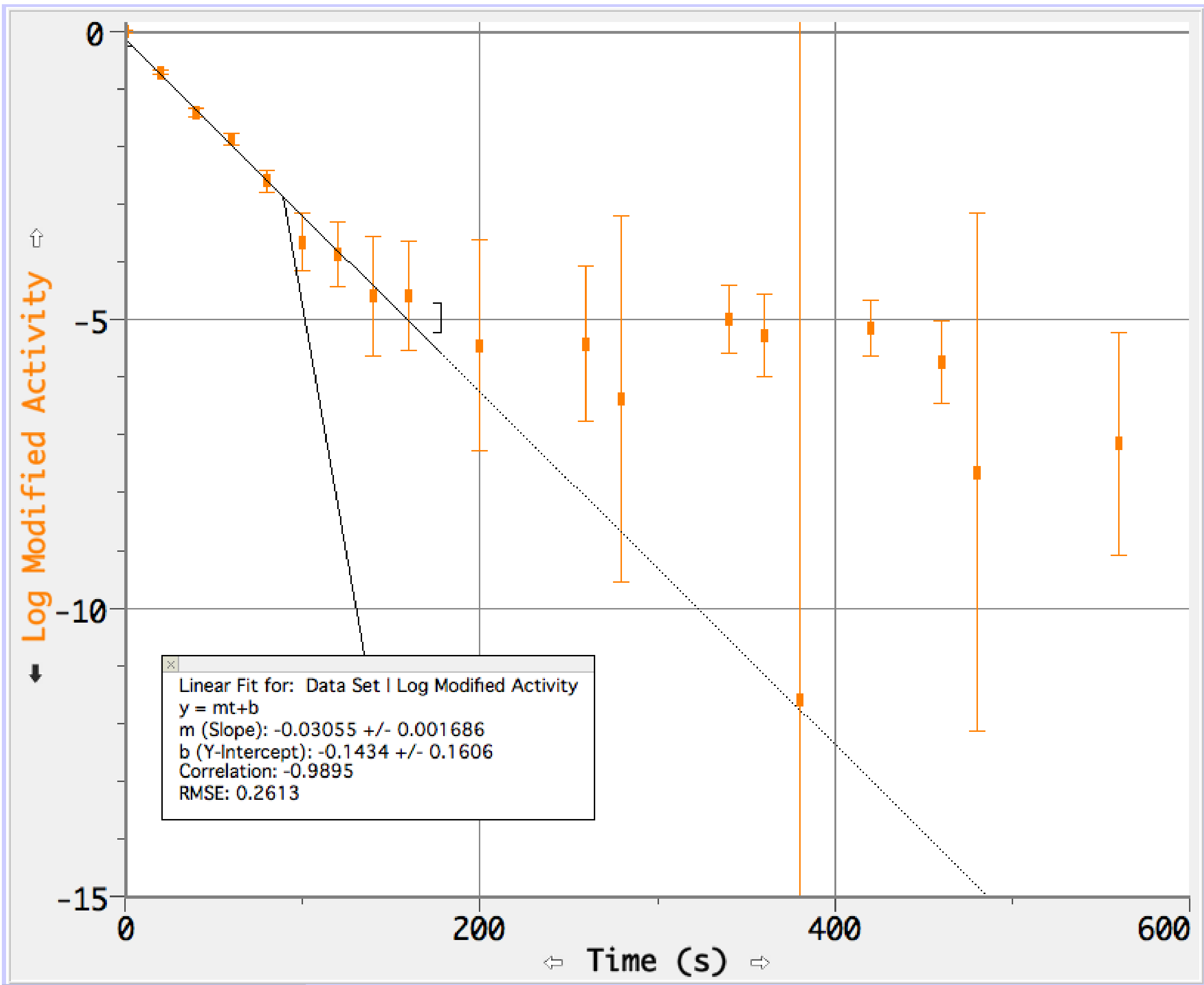

Figure 2: Natural log of Activity (normalized) vs. time (in seconds) from an irradiated silver quarter. The best fit function for the long-lived activity we attribute to Ag 108 has been subtracted from the total. Note that after about 200 s, there are no significant (different from zero) detections. The first 200 s of data therefore represent the activity we can attribute to the fast-decaying isotope (Ag 110), if the hypothesis is not wrong.

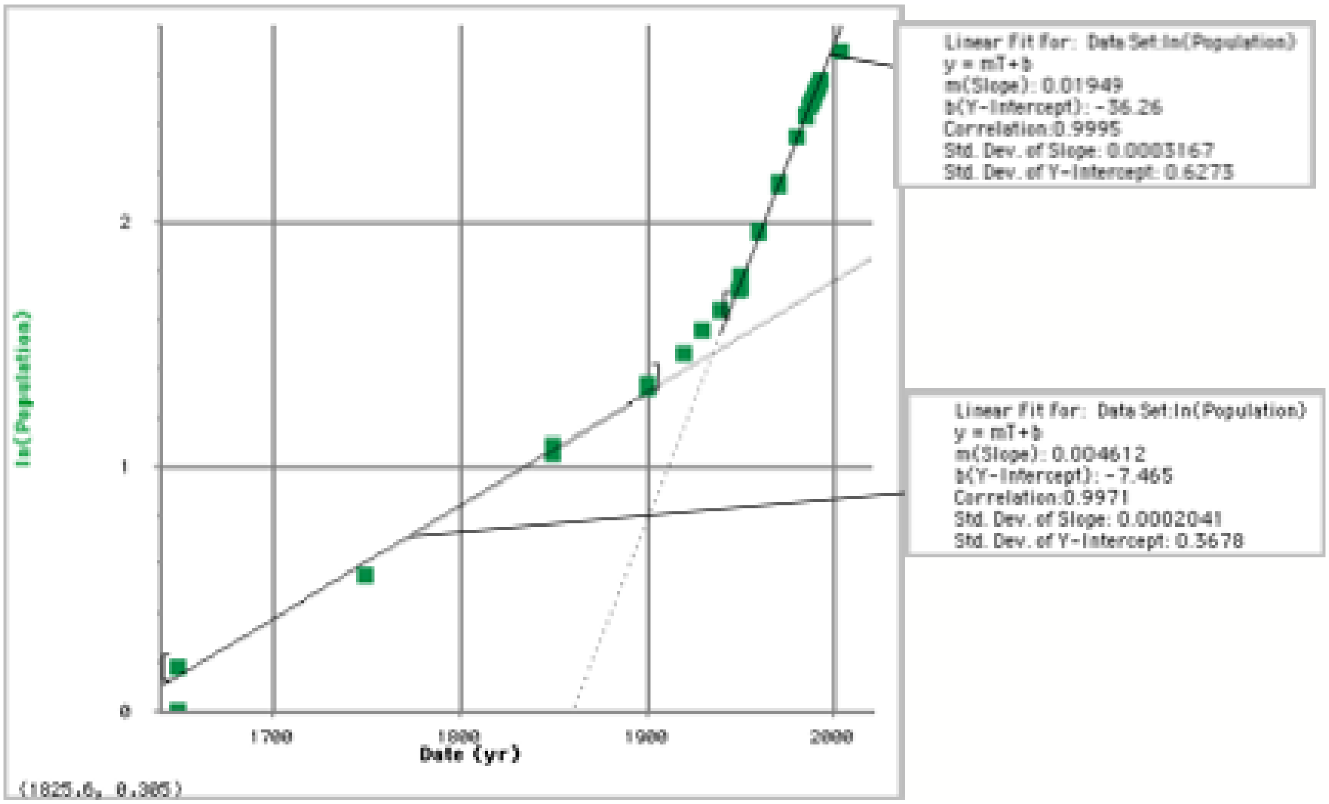

Figure 3: The natural log of world population as a function of calendar year. The data seem consistent with two straight lines. Prior to $\sim$ 1910, the exponential index is $(4.6{\pm}0.2){\times}10^{-3}$ 1/yr, but then it transitions to a much faster rise: $(1.949 \pm 0.003) \times 10^{-2}$ 1/yr.

All of this is to say, you may have heard (and some of you have said, in your homework, as if it's a fact) that population growth is exponential. These data clearly show that this claim is WRONG!!!! It's been two exponentials for the last 300 years or so, until recently, when something else starts happening. That first exponential also clearly cannot be extrapolated back indefinitely.

5. I used Graphical Analysis to examine the relationship between time and both energy and diameter of the accelerators. When I compared time to the natural log of the diameter, I found a clear straight line. However, the log of the energy was clearly not a straight line relationship with time. In order to get a straight line relationship, I had to subtract 1900 from the year. Then when I took the natural log, I got a straight line. These two relationships are shown in Figures 4 and 5, respectively.

Once I had discovered straight lines, I could perform fits. I got the following relationships:

$$\ln\left(E/1.2\text{ MeV}\right) = m_e \ln\left(t - 1900\right) + b_e \tag{15}$$

$$\ln D = m_d t + b_d \tag{16}$$

where D is the diameter of the accelerator in feet, t is the time in years, and E is the energy of the accelerator in MeV. The fit results are $m_e = 16.3 \pm 0.3$, $b_e = -56 \pm 1$, $m_d = 0.23 \pm 0.1$ yr^{-1},

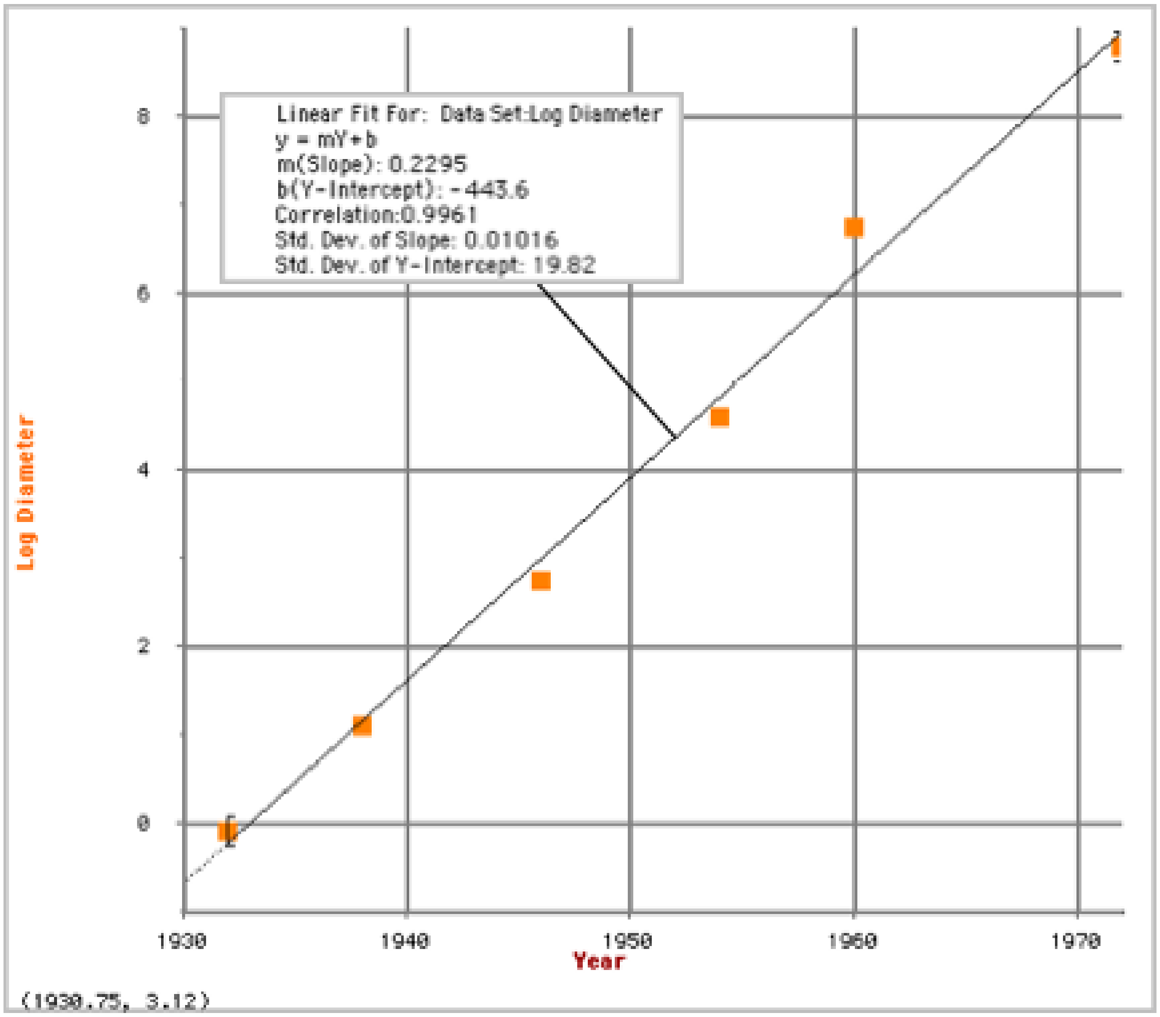

Figure 4: Diameter of accelerators in feet as a function of construction year. The data are consistent with an exponential increase, where $D \propto e^{0.23t}$.

and $b_d = -440 \pm 20$. These two equations can be rewritten in a more "natural" form:

$$E = 1.2 \text{ MeV } e^{b_e} \; (t - 1900)^{m_e} \tag{17}$$

$$D = e^{b_d} \; e^{m_d t} \tag{18}$$

Applying these formulas, I would predict that an accelerator in the year 2009 would have a diameter of $\sim 1 \times 10^6$ km and acheive energies of $\sim 9 \times 10^5$ GeV. At $1 million per GeV, that's nine hundred billion dollars! The LHC is coming on line with a diameter of about 9 km, and the budget is looking to be something on the order of $5 billion, so technology has gotten cheaper since 1973. So that extrapolation is certainly wrong (for one thing, 10^6 km is three times the distance to the moon,so the number is ludicrous on the face of it). The energy they expect to achieve is 1.4×10^4 GeV, which is certainly less than we can extrapolate from 1973, but I am surprised it's not worse!

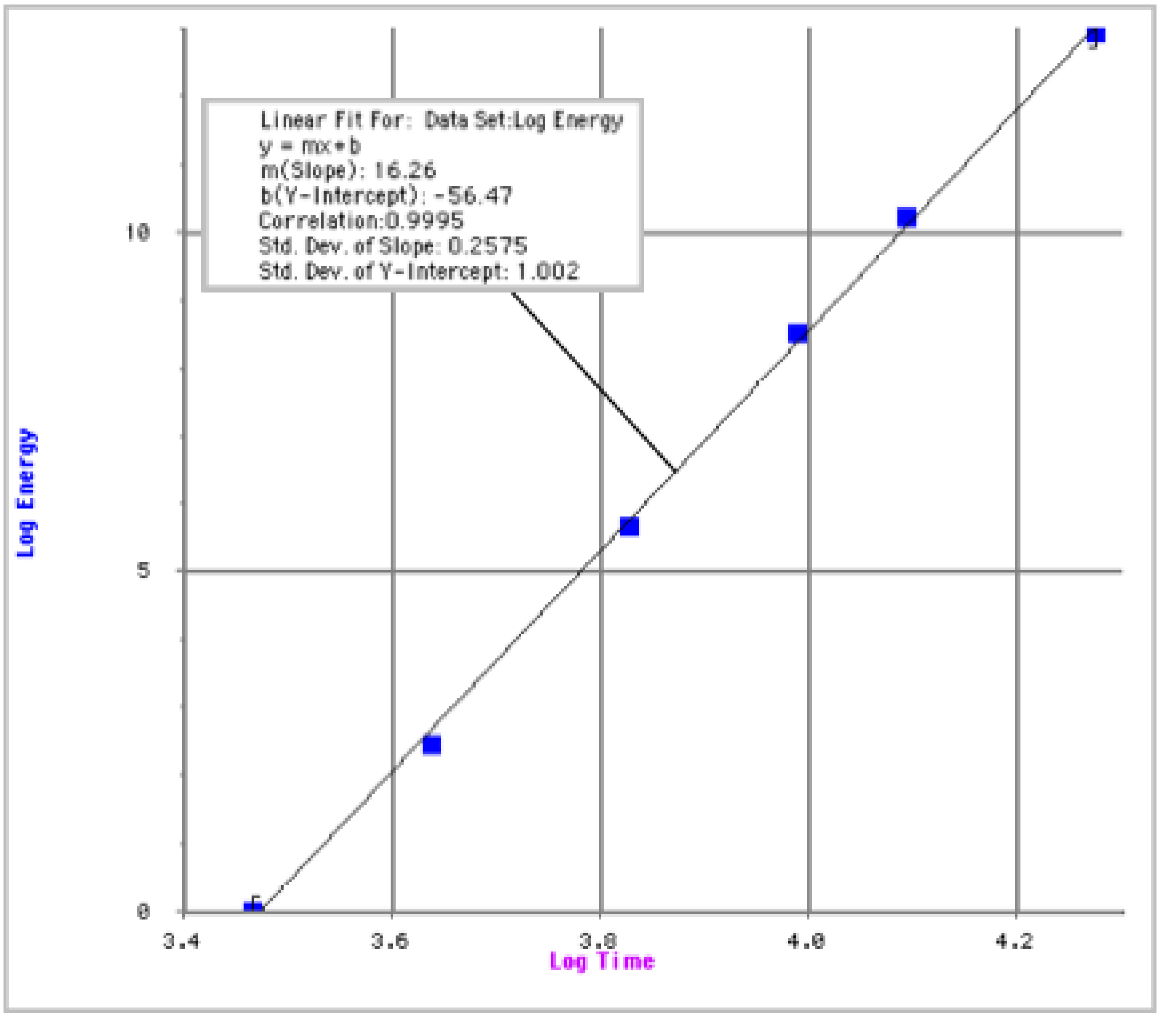

Figure 5: Energy of Accelerator (in MeV) as a function of time (in years) since 1900. The data are plotted on a log-log scale, showing a power-law relationship where $E \propto t^{16}$. That's freakishly fast and cannot be expected to continue indefinitely.