Homework #6: Nuclear Energy

1. ${}_{15}\mathrm{P}^{32}$ decays by emitting an electron. Look up the properties of this decay process using the link to the table of the nuclides that is on the class web page.

a) What is the maximum kinetic energy of the electron that is emitted in this decay process?

The problem says that I am supposed to get the properties of this decay process using the web site table of the nuclides. If I go to the page for ${}_{15}\mathrm{P}$, and click on the P-32 isotope, I see that it is unstable to beta decay.

If I click on the "Beta" link on the web page, I get an energy vs. Z diagram, that shows the change in energy going from ${}_{15}\mathrm{P}^{32}$ to ${}_{16}\mathrm{S}^{32}$ is 1.7103 MeV. It also says below that the maximum energy (this means kinetic energy) of the emitted beta ray (an electron) is 1.7103 MeV. If I don't trust that graph (always wise to double-check), I can calculate the change in rest energies myself:

$$KE = E_{\mathrm{P}} - E_{\mathrm{S}} - E_{\mathrm{e}} = E_{0n} - E_{0p} + BE_{\mathrm{S}} - BE_{\mathrm{P}} - E_{\mathrm{e}}, \qquad (1)$$

that is, the maximum kinetic energy is what's left over when a neutron turns into a proton, and the nuclear binding energy shifts, and you take out the rest energy of the electron. Putting in the numbers:

$$KE = 939.56 - 938.27 + 271.78 - 270.85 - 0.51 = 1.71 \text{ MeV}, \qquad (2)$$

which is not different (to within my significant figures) from the number in the web chart.

b) Use the various conservation laws to determine the physical properties of the daughter nucleus *e.g.* Z number, B number, rest energy.

Since this is a beta decay, the B number of the nucleus remains the same: 32. A neuton has, however, switched to a proton, so the Z number goes up by one to 16, which is sulfur. Since the maximum energy the electron can have is 1.7103 MeV, I can calculate the rest energy of the daughter nucleus through conservation of energy. I know that the net drop in energy through the decay has to be the rest energy of the electron plus the maximum kinetic energy it can have, which is 1.7103 MeV+0.5110 MeV = 2.2213 MeV. So the daughter nucleus must have this much less rest energy than the parent, which implies a rest energy of $15E_{0p}+17E_{0n}-BE_{\mathrm{P}}-2.2213 = 29773.49$ MeV. If I check what the rest energy of ${}_{16}\mathrm{S}^{32}$ actually is, I get $16E_{0p} + 16E_{0n} - BE_{\mathrm{S}} = 29773.49$ MeV. Not bad!

2. Estimate how much energy was released in the first (non-test) atomic bomb explosion? Give answers in MeV, joules, and tons of TNT. (1 kiloton of TNT $\sim$4 TJ = 4×10^{12} J)

We know that the mass of the Uranium-235 material that underwent fission was 0.7 kg. The atomic mass of Uranium-235 is 238.0289 g/mol, so in 700 g, there must be 1.77105×10^{24} atoms (using Avogadro's number). Each nucleus starts out as U^{236} (having absorbed a free neutron) and ends up as two neutrons, a Cs^{140} nucleus, a Zr^{94} nucleus, and three electrons (Sr goes to Y which goes to Zr in two beta decays). I will ignore the energies of the neutrinos. I can therefore calculate how much energy is available from each nucleus's decay.

The initial energy is $Z_U \times E_p + (B_U - Z_U) \times E_n - BE_U$. From this I must subtract $2E_n$ and $3E_{e^-}$, as well as $Z_{Cs} \times E_p + (B_{Cs} - Z_{Cs}) \times E_n - BE_{Cs}$ and $Z_{Zr} \times E_p + (B_{Zr} - Z_{Zr}) \times E_n - BE_{Zr}$. I lost three neutrons and gained three protons and three electrons, and the binding energies shifted. In terms of the energy released, this yields

$$\Delta E = E_f - E_i = 3(E_p - E_n + E_{e^-}) + BE_U - BE_{Cs} - BE_{Zr} \qquad (3)$$

$$\Delta E = 3(938.272029 - 939.565360 + 0.510998) + 1790.415042 - 1164.011653 - 814.676497 \quad (4)$$

$$\Delta E = -2.346999 - 188.273108 \quad (5)$$

which yields 190.620107 MeV released for every U^{236} nucleus that does this. That's a total energy released of 3.71629×10^{26} MeV, or 3.71629×10^{32} eV, which is 5.953×10^{13} J, or 59.53 TJ, or about 15 kilotons. For reference, Wikipedia[1] says "The official yield estimate of "Little Boy" was about 15 kilotons of TNT equivalent in explosive force." If that number comes from a measurement, rather than just doing the same calculation we did, you can see how accurate the calculation is!

3. (a) Using the relevant conservation laws (B, Z, Energy, family number, momentum), write out the formulas for these three reactions and fill in the other particles I haven't mentioned.

First, we need two protons to make a deuterium. That requires a proton changing to a neutron, which means we need to give off a positron and a neutrino.

$$p + p \rightarrow ^2 D + e^+ + \nu_e \quad (6)$$

This balances B, Z, and electron family number. Since the reation is a collision, we can get the extra energy we need from the kinetic energies of the protons.

Now a proton hits the ^{2}D to make ^{3}He. Odds are this will require some momentum to be carried away, and there is definitely an energy excess, but all the other numbers are already conserved, so it has to give off a photon. This point is worth stressing. Because the He will be moving, you need a γ to balance out the momentum. You cannot have a collision make only one particle.

$$p +^2 \text{D} \rightarrow^3 \text{He} + \gamma \quad (7)$$

Finally, the two ^{3}He nuclei come together to make a ^{4}He and two protons. There are six baryons on each side of the equation, and a Z number of four on each side. With three daughter particles, energy and momentum are probably taken care of, so we can keep this one simple and say:

$$^3\text{He} +^3 \text{He} \rightarrow^4 \text{He} + p + p \quad (8)$$

(b) Compare the net change in energy and calculate how many times this process has to happen every second in the sun.

We start out with six protons and end up with a ^{4}He nucleus four protons, and two positrons. So the net change in energy will be

$$\Delta E = E_f - E_i = 2E_p + 2E_e + (2E_p + 2E_n - BE_\alpha) - 6E_p = -2E_p + 2E_n + 2E_e - BE_\alpha \quad (9)$$

Plugging in the numbers, we get a net loss in energy of 24.687 MeV, every time the cycle completes. Note that if we don't include the binding energy, this cycle would need a *gain* in energy, rather than giving off energy! Now, if we were to include that the two positrons go off an annihilate with two electrons somewhere, that would add another ~ 2 MeV to the cycle, but I am going to ignore that, because in the end I only know the luminosity to one sig fig, anyway. The sun gives off 4×10^{26} W, (one sig fig) which is 2.5×10^{45} eV/s, which is 2.5×10^{39} MeV/s. So in order to produce this much energy in a second, we need the cycle to complete 1×10^{38} times.

Just FYI, if you consider that 10% of the sun's mass ($\sim 10^{30}$ kg of mostly hydrogen) is in the core where the fusion is taking place, then there are roughly 10^{29} kg worth of protons available for

[1] The Unerring Font of All Knowledge.

fusion in the center of the sun. That works out to about 5×10^{55} protons. If we lose 4 protons a cycle, and complete 1×10^{38} cycles each second, then it will take about 1.3×10^{17} s for the sun to run out of protons, which is about 5 billion years. So, nothing we need to worry about. Global warming will kill us long before that happens.

4. After the excited Barium-137 nucleus gives off its gamma ray (that you measured in the half-life lab), what is the recoil speed of the nucleus?

There is an easy way and a hard way to do this. The easy way is to claim that since the nucleus has *so* much more rest energy than the energy of the photon, its kinetic energy will be negligible, so you can claim that *all* of the $\Delta E = 0.661$ MeV goes to the photon. In that case, the total momentum before and after must be zero, so the momentum of the nucleus must be equal and opposite to the momentum of the photon. The rule we said in class was that the momentum of a photon is its energy over c, so that gives us:

$$v = \frac{E_\gamma}{mc} = \frac{\Delta E}{mc}. \tag{10}$$

Now if I divide both sides by c, I get

$$\beta = \frac{v}{c} = \frac{\Delta E}{mc^2} = \frac{\Delta E}{ZE_p + (B - Z)E_n - BE}, \tag{11}$$

where the denomenator is now the rest energy of the nucleus. We use β to represent the speed in dimensionless units. So the ratio of the energies tells you the recoil speed! Plugging in the numbers, I get $v/c = 0.661/127498 \sim 5.2 \times 10^{-6} \rightarrow v \sim 1.6 \times 10^3$ m/s. Don't worry, the barium nucleus won't go flying away – it will be stopped very quickly by the other atoms in the vicinity.

Now, you may wonder about my bold assertion that the energy that goes into the photon is *all* of ΔE. Well, consider that the kinetic energy of the nucleus is given by $mv^2/2$, which can also be written as $E_0\beta^2/2$. We know that $\beta \sim 10^{-6}$, so $\beta^2 \sim 10^{-12}$. For a rest energy of something like 10^5 MeV, that means the kinetic energy would be something like 10^{-7} MeV, which is much, much, much less than ΔE, thus justifying my assertion.

5. How does a ball thrown in the air not violate the law of conservation of momentum? Well, you can answer this two ways: either you can look at the "if", which says *if* no external force acts on a system, *then* momentum is conserved. If your system is the ball, there is an external force on the system, so the law does not apply. Alternately, you could define your system to be the ball *and* the earth, in which case there is no external force and the law *does* apply. As the momentum of the ball changes, the momentum of the earth changes to cancel it out. The total momentum of the ball/earth system remains zero.

What you *cannot* say is that since the ball comes back down with the same speed, its final momentum cancels out its initial momentum so the total momentum stays zero. There are at least three problems with that. First, you have chosen "special times" at which you think the law holds – if it's a law, it needs to hold for *all* times, not just two particular ones. A broken clock is right twice a day – it's still broken. Second, if the momentum is *conserved*, it needs to stay the same. The momentum at the beginning is upwards, the momentum at the end is downward. That is *not* the same. Thirdly, you cannot *add* before and after, you have to set before and after *equal* to each other. Any way you slice it, this approach just does not work.

6. We're looking how to get Pu^{239} from U^{238}. First, you add the slow neutron to get U^{239}. According to the chart, this isotope is unstable and decays to Np^{239}. This is a beta decay. Np^{239} then undergoes

a beta decay itself to become Pu^{239}. We therefore start with U^{239} and end up with Pu^{239} and two electrons (we consider the neutrinos to be negligible). To get the amount of energy released we subtract before from after to get

$$\Delta E = E_{\rm Pu} + 2E_e - E_{\rm U}. \tag{12}$$

Now, in the beta decays, we lose two neutrons and gain two protons. The other protons and neutrons stay there both before and after, so they cancel. This equation therefore simplifies to

$$\Delta E = 2(E_p + E_e - E_n) - BE_{\rm Pu} + BE_{\rm U}. \tag{13}$$

We can look up these numbers to get

$$\Delta E \text{ (in MeV)} = 2(938.272029 + 0.5109989 - 939.565360) - 1806.921454 + 1806.500907 = -1.98535. \tag{14}$$

Doesn't seem like much, but any macroscopic amount of U is going to have something like 10^{23} nuclei in it, so you can get a *lot* of energy out.