Homework #2: Quantitative Reasoning, Units, and Dimensions

1. **Setup**: To estimate the fraction of the surface area of the continental United States that is covered by motor vehicles, I need to estimate the surface area of the US, the surface area of a typical motor vehicle, and the total number of vehicles in the US. **Work:** I know that the population of the US is about 3×10^8 people. I will assume that accounting for the number of people in Hawaii and Alaska will not significantly decrease the total. Although not every person in the US has a vehicle, many families have more than one, and there are many car lots and junk yards that have lots of vehicles sitting around, so I think 1 vehicle per person is pretty good for an order of magnitude. I would be surprised if it were off by a factor of ten. These days, people seem to be driving large trucks as much as normal-sized cars, unfortunately, so I will estimate the surface area of a vehicle to be about 10 m^2. It's probably less than that, but there are also semi-trucks and the like to take into account, so I will err on the large side. That makes about 3×10^9 m^2 of vehicular area. If I approximate the US as a rectangle, about 6500 km (4000 miles) across, then it looks to be about 2/3 as high as it is wide, so that's about 3×10^7 km^2, or 3×10^{13} m^2 for the area of the continential US[1]. **Answer:** Dividing these two yields 10^{-4}, or 0.01%. **Comment:** This is about the size of the state of Connecticut.

2. **Setup:** I am asked to estimate the volume of trash (in m^3) produced in the US in a year. I assume we're talking about household trash, not industrial trash or construction debris. I will use my household as representative, and try to extrapolate to the whole nation. **Work:** My wife and I pretty regularly put out one kitchen trash bag of trash each week. The kitchen trash can is about 0.8 m tall, 0.3 m wide, and 0.4 m deep. That yields 0.096 m^3, for the two of us, so that's 0.05 m^3 per person per week. There are 52 weeks in a year, and 3×10^8 people in the country, so if my wife and I are representative, (**Answer:**) that's 8×10^8 m^3 of trash per year. We're probably not representative (our neighbors across the street have a family of four, and their trash bin is usually overflowing each week), so probably something like 10^9 m^3 is a pretty good guess. **Comment:** That's a *billion* cubic meters of trash! Spread it a meter deep, and it will fill a hundred million square meters, that's 25 thousand acres!

3. **Setup:** This is an exercise in significant figures and scientific notation.

Work and Answers:

a) add the exponents to get 10^1 (not 10!)

b) Subtract the exponents to get 0.5×10^{16}, which in proper scientific notation is 5×10^{15} (one significant figure).

c) This is multiplication, and two of the numbers only have one sig fig, so the answer should have one sig fig: 6×10^7.

d) This is addition, so you need to make sure the decimals line up: 3×10^{-8} is the same thing as 0.3×10^{-7}, so this should be written $(3 + 0.3) \times 10^{-7}$. That means the tenth place (the first place after the decimal is "three plus don't know", so the answer is "don't know", and the final result remains 3×10^{-7}. (You can think of it this way: adding a second number that is smaller than the uncertainty in the first number does not significantly change the first number.)

[1] According to Wikipedia, the unerring font of all knowledge, the surface area of the US is about 9.6×10^6 km^2, so I am in the right ballpark.

Comment: No real deep insight here, but it's practice for how to keep track of what you know, and a few of the results are surprising if you don't think carefully, like adding a smaller number that makes no effective change in part d.

4. **Setup:** This problem involves using dimensional analysis to get a sense of relative scales in the universe. We are asked to compare atomic dimensions with macroscopic dimensions.

Work, Answers, and Comments:

a) The diameter of the nucleus is about 10^{-14} m, so there are 10^{14} nuclei in a meter (if you could lay them end-to-end). At one nucleus per second, this would take 10^{14} s, which is something like 3 million years. That's a really long time.

b) Volume goes like the size cubed, so the cube of 10^{-10} m is 10^{-30} m^3. Divide by the cube of the size of the nucleus (10^{-42} m^3), and you get 10^{12} nuclei per atom. Put another way, the nucleus only takes up one-trillionth of the volume of the atom!

c) A large star is about about 10^{12} m across. The Earth is about 10^7 m across. So about one-hundred thousand (10^5) Earths could fit across the span of a large star. (note that is *diameter*, not *volume*! You could fit a thousand trillion Earths into the volume of a large star.) The sun is very big.

Note that for each problem, I've compared the answer to everyday experience, so that you get a sense of scale, and the problem becomes more than just number-crunching.

5. **Setup:** This problem is simply for practice in unit conversion and significant figures.

Work, Answers, and Comments:

(a) 186,000 mile/sec is 3.00×10^8 m s^{-1}. :-) Seriously, a mile is about 1600 m, so if I multiply 186,000 by 1,600, I get 2.976×10^8, which to three significant figures is 3.00×10^8 m s^{-1}. That's the SI version of the speed of light in a vacuum.

(b) 1 yard. A yard is 36 inches, and an inch is 2.545 cm. So 1 yard = 0.9 m. (To one sig fig.) Not much more to say about that one.

(c) 32 ft/sec^2. I happen to know that's the acceleration of Earth's gravity near the surface of the Earth, so I should get 9.8 m s^{-2}. A foot is 12 inches, and an inch is 2.545 cm, so 32 ft = 9.7728 m = 9.8 m, to two sig figs. Which is indeed what I expected, since I know that 9.8 m/s^2 is the acceleration due to Earth's gravity at its surface, in SI units.

6. **Setup:** I am asked to find the dimensions of energy per unit time, and I should also give the SI unit of this quantity. **Work and Answers:** The dimensions of energy are mass times length squared divided by time squared. So the dimensions of power are mass times length squared divided by time cubed. Energy per unit time is also known as "power." The SI unit of power is the watt (W). **Comment:** This problem serves to emphasize the difference between a dimension and a unit.

7. **Setup:** This problem asks which of a series of equations are dimensionally consistent. Being able to check if an answer is dimensionally consistent is a *very* important skill to cultivate. I will say that an equation "works" if each side of the equals sign has the same dimensions once you cancel out redundant powers. If the two sides of the equals sign have different dimensions, they are not, in fact, equal, and I will say the equation does not work. Often, when you solve a problem, you can catch a mistake by quickly checking to make sure your answer has the right dimensions (the "unit gorilla"). **Work and Answers:** Let's go through them one by one:

a) time = distance divided by speed, which is $\ell/(\ell/t) = t$ =time, so that's okay.

b) energy is mass times speed squared, so energy squared over mass squared would be speed to the fourth power. No, this does not work.

c) speed times time squared would be distance times time, not distance. This does not work. (the 1/2 is dimensionless.)

d) mass times time times speed times distance yields mass times distance squared (the time cancels out the time in the denominator from the speed, which means the speed contributes an extra distance dimension.). This is not the same dimension as energy.

e) speed times time yield distance. This works.

f) distance over time is speed, which is then squared and multiplied by mass – those are the dimensions of energy, so that works.

g) speed over time is distance over time squared, not time, so this does not work.

h) If you divide both sides by mass squared, you get back something a lot like part b of this problem, but since it's speed to the fourth, this one is correct, where part b is wrong.

Comment: Note that dimensions are not the same thing as units, although since any dimension must have *some* unit associated with it, you can do this kind of analysis with units. However, since you can switch units around (like measuring length with units of seconds), you have to be careful. You could get a system of units where m = s, and it would not be wrong, because both sides of the equation would have *dimensions* of length. See the next problem for an example.

8. **Setup:** I am asked to find three lengths in a system of units where the speed of light is 1. **Work:** If the speed of light is 1, I can express lengths in units of seconds by calculating how much time it would take light to traverse that length. In essence, since speed is distance by time, and since any number divided by one is unchanged, I can change meters into seconds by dividing by the speed of light (which in these units is 1). In order to get the units to work out, I have to use the conversion that 1 (in these units) = 3.00×10^8 m/s in SI units. Therefore:

Answers:

a) my height in SI units is 1.93 m, or 6.43×10^{-9} s in the new unit system (light moves about a foot per nanosecond, so at 6.3 feet, I would expect to get something like 6.3 ns, and 6.4 ns is certainly close enough!).

b) Since light moves at 186,000 miles per second, one mile in Imperial units is 1/186,000 s in the new units, or 5.38×10^{-6} s.

c) One light year, if light moves at the speed 1, is simply one year in time units. There are approximately 3.1×10^7 s in one year, so this is how many seconds in a light year, if $c = 1$.

Comment: This problem is meant to emphasize the arbitrariness of units, and how they change as you change systems of units. Even the same dimensions can be written in completely different units, if you choose your system carefully.

9. **Setup:** If a fast electron is moving at 10^8 m/s, then to find out how long it will take to go various distances, I need merely to divide the distance by the speed. Therefore, for the electron to go...

Work and Answers:

a) across a nucleus: 10^{-14} m/10^8 m/s=10^{-22} s. That's a very short time. If the electron is going to interact with the nucleus at all, the interaction will have to take place in that amount of time.

b) across an atom: 10^{-10} m/10^8 m/s=10^{-18} s. That's ten-thousand times longer than the time it takes to cross the nucleus, but still a very short time.

c) a giant molecule: 10^{-7} m/10^{8} m/s=10^{-15} s. Again, that's a thousand times longer than the atom-crossing time, but still shorter than just about any time comprehensible to the human brain.

Comment: Think about that! Even though the electron is booking along at nearly the speed of light, it still takes 10 million times longer to go through a molecule than it does to go through a nucleus!!! That nucleus is *really* small!